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SOCIAL UTILITY FUNCTIONS FOR STRATEGIC DECISIONS IN PROBABILISTIC VOTING MODELS*

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This paper studies societies which have probabilistic voting that is smooth, scalable and unbiased. Its results establish that, in such societies, the decisions of vote-seeking candidates who start at a common location (such as the status quo for the society's policies and/or the same allocation of campaign resources) contain implicit rationality properties. In particular, it shows that in every such society there exist social utility functions which simultaneously rationalize the directional Nash behavior of candidates, the stationary electoral equilibria, and the non-degenerate local electoral equilibria which can occur at these locations. This is shown to be true both for unconstrained and for constrained sets of possible candidate locations. An example of such a utility function (which occurs in every one of the societies under consideration) is also provided.

Key words: Probabilistic voting model; social utility function; directional Nash equilibrium.

1. Introduction

Most of the work which has been done on the economic theory of majority rule and electoral competition has been concerned with the question: Do vote-speeking decisions by political candidates exhibit 'consistency' or 'rationality' properties (such as the existence of equilibria or the possibility of being rationalized by a utility function)? From the existing literature on this topic, it is clear that when (1) each voter has a preference ordering, (2) any particular voter will vote for a candidate whenever he strictly prefers that candidate's platform to the one offered by his rival, (3) indifferent voters are split evenly, and (4) there are two candidates who both know the preferences of every voter, the answer is generally, 'no'. Indeed, this answer has been shown to be appropriate whether candidates are making directional, local or global strategic decisions (Note 1; see section Notes at the end of this paper). However, it is not so clear that we should be content with the answers that

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have been derived from the 'deterministic voting models' which satisfy (1)–(4). After all, one of their central assumptions (namely (4)) is certainly not descriptive of actual elections. In addition, alternative models have been developed in which candidates *do* have uncertainty about voters' decisions. In them, the assumption of complete information about voter's behavior has been replaced by the assumption that, at each pair of candidate strategies, the candidates have likelihoods for the events "will vote for candidate 1", "will vote for candidate 2" and "will abstain" for each voter (or for individuals drawn randomly from each group of voters in some partition of the population). Of course, these 'probabilistic voting models' still fall short of being complete economic models of the electoral process – but they *have* taken an important step in that direction. In addition, a number of authors who have taken this step, has found that these alternative models lead to conclusions that are contrary to the ones which deterministic voting models lead to. In particular, many of them have found that, in their models, candidate decisions have certain rationality properties (especially with respect to equilibrium existence questions) which are absent in corresponding deterministic voting models. References where this has occurred include Hinich and Ordeshook (1969, 1971), Hinich, Ledyard and Ordeshook (1972, 1973), Brams and Davis (1973, 1974), Shane (1977, Section 3), Hinich (1977, 1978), Brams (1978, Chapter 3) and Coughlin and Nitzan (1981a, 1981b) (Note 2). In this paper it will be shown that the phenomenon of rationality for candidate decisions in probabilistic voting models is, in fact, far deeper than has been indicated in any of these earlier articles.

2. Probabilistic voting models

$X \subset R^n$ will be an open, star-shaped (Note 3) set of possible candidate 'locations'. These could be policy proposals, campaign resource allocations, or vectors which specify both of these. $C = \{1,2\}$ will be an index set for the two candidates. Whenever we are considering a particular pair of directions or locations for the candidates the elements in the pair will always be ordered according to these indices. Individual voters will be indexed by elements, a, in a set $A \subset R^l$. One possible interpretation is that $A = X$ and each index, a, is an 'ideal point' for the voter. Another possibility is that $a = 1, \ldots, |A|$ (where $|A|$ is the cardinality of the finite set A, and $l = 1$). The candidates' expectations about the voters' behavior will be summarized in 'probabilistic voting functions',

$$P_a^i : X \times X \to [0,1] \quad \text{for } i = 0,1,2 \text{ and } a \in A. \tag{1}$$

$P_a^i(\psi_1, \psi_2)$ with $i \in C$ denotes the likelihood that an individual who is randomly drawn from the citizens indexed by 'a' will vote for candidate i when the two candidates are at the locations ψ_1 and ψ_2, respectively. This could be an objective likelihood or a subjective likelihood that is believed by both candidates.

$$P_a^0(\psi_1, \psi_2) = 1 - P_a^1(\psi_1, \psi_2) - P_a^2(\psi_1, \psi_2)$$

is the probability that such an individual chooses to abstain. By starting with aggregate assumptions in this way, we are not restricted to particular assumptions about individual voting behavior (such as those in Hinich and Ordeshook (1969), Hinich, Ledyard and Ordeshook (1972, 1973) or Coughlin and Nitzan (1981a, 1981b), for instance). The population of individuals will be summarized by a probability measure space, (A, Σ, μ). We will also assume that, for each $i \in C$ and $(\psi_1, \psi_2) \in X^2$, $P_a^i(\psi_1, \psi_2)$ is integrable with respect to (A, Σ, μ). When all of the above assumptions are satisfied, we will say that we have 'a society with probabilistic voting'.

In the analysis in this paper, three additional assumptions (which have appeared in the probabilistic voting models cited in the introduction) will also be included. First, we will assume that, for each $a \in A$ and $i \in C$, there exists functions (Notes 4–7) $G_a: X \to R^1$ and $\bar{P}_a^i: R^2 \to R^1$ such that

$$P_a^i(\psi_1, \psi_2) = \bar{P}_a^i(G_a(\psi_1), G_a(\psi_2)) \quad \text{for each } (\psi_1, \psi_2) \in X^2. \tag{2}$$

This will be called 'scalable' probabilistic voting (following the terminology in Calfee (1979), for instance). Second, we will assume that

$$P_a^1(x, y) = P_a^2(y, x) \quad \text{for every } a \in A \text{ and } (x, y) \in X^2.$$

This will be called 'unbiased' probabilistic voting (since it is a direct extension of the assumption about voting behavior that was given this label in Hinich, Ledyard and Ordeshook (1973, p. 167)). Third, we will assume that, for each $a \in A$ and $i \in C$, the functions $\bar{P}_a^i(\cdot)$ and $G_a(\cdot)$ (and hence $P_a^i(\psi_1, \psi_2)$) are twice continuously differentiable functions of their arguments in some neighborhood of (ψ_1, ψ_2) whenever $\psi_1 = \psi_2$. We will also assume that all of the relevant first and second partials of these functions are integrable with respect to (A, Σ, μ) in the same neighborhoods. This will be called 'smooth' probabilistic voting (Note 8).

The third assumption is one which is satisfied in the probabilistic voting models in Hinich, Ledyard and Ordeshook (1972, 1973), Brams and Davis (1973, 1974, 1982), Shane (1977, Section 3), Hinich (1977, 1978), Brams (1978, Chapter 3) and Coughlin and Nitzan (1981b). Furthermore, it is an assumption which is natural when candidates are estimating voters' behavior. However, it should also be made clear that this assumption fully separates the models which are analyzed here from those which have been studied in the deterministic voting literature. This follows from the fact that, when voting is deterministic (i.e., assumptions (1)-(4) in Section 1 are satisfied), there are 'fundamental discontinuities' in the candidates' payoff functions. For a thorough discussion of these discontinuities, see Kramer (1978b).

3. Directional Nash behavior

It will be assumed throughout that, when they are making strategic decisions, the candidates are interested only in the effects of their actions on their expected

pluralities - this being a feature which is common to all of the articles cited in Section 1. When we have a society with probabilistic voting, these effects can be determined by studying the candidates' expected plurality functions:

$$\mathrm{Pl}_1(\psi_1,\psi_2)=\int_A(P_a^1(\psi_1,\psi_2)-P_a^2(\psi_1,\psi_2))\,d\mu(a) \tag{3a}$$

and

$$\mathrm{Pl}_2(\psi_1,\psi_2)=\int_A(P_a^2(\psi_1,\psi_2)-P_a^1(\psi_1,\psi_2))\,d\mu(a)=-\mathrm{Pl}_1(\psi_1,\psi_2) \tag{3b}$$

for $(\psi_1,\psi_2)\in X^2$. In this paper we will be specifically analyzing all of the possible situations which can arise in which the candidates start at the same location $x\in X$ (i.e., $\psi_1=\psi_2=x$, where x is the status quo for the societies policies and/or the same allocation of campaign resources, for instance), and each one of them chooses a 'best' direction in which to vary his location.

Since X is open, at each $x\in X$ the set of possible directions will be

$$T(x)=\{u\in R^n:\|u\|=1\}\cup\{\mathbf{0}\in R^n\}. \tag{4}$$

The particular behavioral assumption that we will use to specify our notion of 'best' will be the Nash assumption that each candidate will choose a direction which maximizes his marginal payoff, when his rival's location is taken as fixed (e.g., see Moulin (1979)). This implies that, at any given location $x\in X$, candidate 1 will choose

$$\left[\frac{\partial\mathrm{Pl}_1(\psi_1,x)}{\partial\psi_{1h}}\Big/\left\|\frac{\partial\mathrm{Pl}_1(\psi_1,x)}{\partial\psi_{1h}}\right\|\right]_{\psi_1=x}\quad(h=1,\ldots,n) \tag{5a}$$

when $[\partial\mathrm{Pl}_1(x,x)/\partial\psi_{1h}]\neq\mathbf{0}$, and (similarly) candidate 2 will choose

$$\left[\frac{\partial\mathrm{Pl}_2(x,\psi_2)}{\partial\psi_{2h}}\Big/\left\|\frac{\partial\mathrm{Pl}_2(x,\psi_2)}{\partial\psi_{2h}}\right\|\right]_{\psi_2=x}\quad(h=1,\ldots,n) \tag{5b}$$

when $[\partial\mathrm{Pl}_2(x,x)/\partial\psi_{2h}]\neq\mathbf{0}$. When $[\partial\mathrm{Pl}_i(x,x)/\partial\psi_{ih}]=\mathbf{0}$, each direction leads to a marginal payoff of zero. Therefore, when this happens, candidate i ($i=1,2$) can choose any $u\in T(x)$. Having recognized this, it becomes possible to use our assumption that the probabilistic voting in the society is unbiased to simplify our characterization of the candidates' decisions even further. In particular, by (3),

$$\mathrm{Pl}_1(y,x)=\int_A(P_a^1(y,x)-P_a^2(y,x))\,d\mu(a)\quad\text{for any } x,y\in X. \tag{6}$$

Therefore, since there is unbiased probabilistic voting,

$$\mathrm{Pl}_1(y,x)=\int_A(P_a^2(x,y)-P_a^1(x,y))\,d\mu(a)=\mathrm{Pl}_2(x,y). \tag{7}$$

Consequently,

$$\left[\frac{\partial\mathrm{Pl}_1(\psi_1,x)}{\partial\psi_{1h}}\right]=\left[\frac{\partial\mathrm{Pl}_2(x,\psi_2)}{\partial\psi_{2h}}\right]\quad\text{at } \psi_1=\psi_2=x.$$

Hence, the set of best directions for candidate 1 will always be exactly the same as the set of best directions for candidate 2. Therefore, we can summarize the possible strategic choices by the 2 candidates by using the correspondence,

$$H(x)=\begin{cases}\dfrac{1}{\gamma_1(x)}\left[\dfrac{\partial \mathrm{Pl}_1(\psi_1,x)}{\partial \psi_{1h}}\right]_{\psi_1=x} & \text{when } \gamma_1(x)=\left\|\left[\dfrac{\partial \mathrm{Pl}_1(\psi_1,x)}{\partial \psi_{1h}}\right]\right\|_{\psi_1=x}\neq 0,\\ T(x) & \text{when } \gamma_1(x)=0,\end{cases} \tag{8}$$

which simply summarizes these choices for candidate 1. This correspondence will be referred to as 'the directional Nash behavior of candidates'. In passing, it should be mentioned that, while it has been explicitly derived for 2-candidate competitions, it also summarizes the directional strategies that expected plurality maximizing candidates would adopt against fixed incumbents (who, themselves, are forced to defend the status quo). This can be seen directly from (8) itself.

4. Social utility functions for directional Nash behavior

In this section we will focus on the question: Is there an underlying 'social utility function' which rationalizes the directions which candidates would choose at the various possible locations in X (or, phrasing the same thing a little more succinctly, 'rationalizes the directional Nash behavior of candidates')? To say 'yes' to this question will mean (Note 9) that there exists a real-valued, totally differentiable function $\pi(x)$ on X which has the property that, at any given location in X, a direction is a marginal utility maximizing direction if and only if it also a best direction for the candidates at that location. The marginal utility maximizing directions (for a given $\pi(x)$) can, of course, be summarized by the correspondence

$$J(x)=\begin{cases}\dfrac{1}{\gamma_2(x)}\nabla\pi(x) & \text{when } \gamma_2(x)=\|\nabla\pi(x)\|\neq 0,\\ T(x) & \text{when } \gamma_2(x)=0.\end{cases} \tag{9}$$

Then, using this notation, we can restate the rationality requirement on the directions chosen by the candidates as, simply, the existence of a (real-valued, totally differentiable function) $\pi(x)$ such that $H(x)=J(x)$.

The first result of this paper is the following.

Theorem 1. *Suppose that we have a society with probabilistic voting which is smooth, scalable and unbiased. Then there exists a social utility function which rationalizes the directional Nash behavior of candidates.*

The proof can be found in Appendix A.

This theorem establishes that, when we use the probabilistic voting models

specified in Section 2 (or any special case thereof) to study majority rule, we are led to the conclusion that elections lead to rational collective decison-making (in the sense made precise at the beginning of this section). This implication is rather important. In particular, Theorem 1 raises some serious questions about whether the primary conclusion of much of the literature on majority rule (namely that, in general, majority rule leads to strategic decisions which are not collectively rational) is, in fact, descriptive of the world. In addition, this challenge is a rather compelling one – especially when it is recognized that it has been derived from models that were originally developed to provide alternative mathematical descriptions of elections with important empirical characteristics that had previously been left out of voting models.

5. Nash points and local electoral equilibria

Theorem 1 can be used to study equilibria in which neither candidate has an incentive to alter his basic location (in addition to providing insight into the collective rationality properties of candidates' strategic decisions at all possible locations). This, of course, is of special interest because of the fact that much of the research that has been done on voting models has concentrated on studying such equilibria. The first type of equilibrium in this category is the 'Nash point' (e.g., see Moulin (1979)). Stating the relevant definition formally, we say that a particular $x \in X$ is 'a location where a Nash point occurs' if and only if

$$\left[\frac{\partial \mathrm{Pl}_1(x,x)}{\partial \psi_{1h}}\right] = \left[\frac{\partial \mathrm{Pl}_2(x,x)}{\partial \psi_{2h}}\right] = 0 \in R^n. \tag{10}$$

Saying that there is a 'social utility function which rationalizes the locations where Nash points occur' means that there is a real-valued, totally differentiable function $\pi(x)$ which is such that (for any $z \in X$) there is a Nash point at $z \in X$ if and only if z is a stationary point of $\pi(x)$. Therefore, Theorem 1 directly implies the following (taking its premise as given).

Corollary 1. *There exists a social utility function which rationalizes the directional Nash behavior of candidates and the locations where Nash points occur.*

Locations at which both candidates are able to achieve a local maximum are also of special interest (e.g., see Coughlin and Nitzan (1981b) and the related reference cited there). When, in particular, z is a location where there are local maxima which can be identified by using the first and second order information in the candidates' payoff functions at the location (i.e., $z \in X$ is a location where a Nash point occurs and both

$$[\partial^2 \mathrm{Pl}_1(\psi_1, z)/\partial \psi_{1h} \partial \psi_{1k}]_{\psi_1 = z} \quad \text{and} \quad [\partial^2 \mathrm{Pl}_2(z, \psi_2)/\partial \psi_{2h} \partial \psi_{2k}]_{\psi_2 = z}$$

are negative definite on the set $T(z)-0$), we will say that there is a 'non-degenerate local electoral equilibrium at z'. As it turns out, all such equilibria are also rationalized by some of the utility functions whose existence has been established in Theorem 1. This assertion can be made precise as follows. First, saying that there is a 'social utility function which rationalizes the non-degenerate local electoral equilibria for the candidates' means that there is a real-valued, totally differentiable function $\pi(x)$ which has the property that (for each $z \in X$) there is a non-degenerate local electoral equilibrium at the location z if and only if z is a non-degenerate local maximum of $\pi(x)$. Second, using this definition we can state the following.

Theorem 2. *Suppose that we have a society with probabilistic voting which is smooth, scalable and unbiased. Then there exists a social utility function which rationalizes the directional Nash behavior of candidates and the non-degenerate local electoral equilibria for the candidates.*

The proof of this theorem can be found in Appendix A.

The third basic type of electoral equilibrium which has been extensively studied is the global electoral equilibrium (that is, a location where neither of the candidates can improve his expected plurality by moving to any alternative location in X while his opponent remains fixed). As in all studies of non-cooperative games, the existence of such equilibria will follow from the presence of certain concavity and convexity properties in the candidates' payoff functions. However, since we are concentrating on societies with smooth probabilistic voting, when such (concavity or convexity) based equilibria exist, they can then be studied as Nash points or as local electoral equilibria (as in Couglin and Nitzan (1981a) or Hinich, Ledyard and Ordeshook (1972, 1973), for instance). Therefore, results for such equilibria have already been implicitly provided by Corollary 1 and Theorem 2.

6. Constrained equilibria

All of the results which have been derived so far are ones which have assumed that, at each location in X, the candidates can move in any direction. However, it is quite possible (in a particular election) for the feasible locations to be restricted to some subset of X by inequality constraints, $g_\alpha(x) \leq 0$ $(\alpha = 1, \ldots, p)$, and/or equality constraints, $g_\beta(x) = 0$ $(\beta = p+1, \ldots, m)$, that are twice continuously differentiable. When there is such a subset, it will be denoted by $S \subset X$. For instance, if X is a set of possible policy proposals, S may be the set of policies which are actually feasible at the time of a particular election (e.g., as in Hinich, Ledyard and Ordeshook (1972, 1973) and Coughlin and Nitzan (1981a, 1981b)). Alternatively, if X is a set of possible campaign resource allocations, S could be the set of those allocations that are feasible for a given campaign fund (as in Brams and Davis (1973, 1974), Shane (1977, Section 3) and Brams (1978, Chapter 3)). Analogous

possibilities also exist when X consists of vectors which specify both policy proposals and campaign expenditures. Interestingly, even when the candidates' locations are restricted in this way, their behavior will still be rationalized by the utility functions whose existence has been established in Sections 4 and 5. In the remainder of this section the exact sense in which this is true will be made clear. Most of the definitions which are used in this discussion are drawn directly from Coughlin and Nitzan (1981b). In stating results the premise of Theorem 1 will be assumed.

When $s \in \text{int}(S)$, the set of feasible directions for the candidates, $T(s)$, is given by (4). When $s \in \partial(S)$, the set of feasible directions for the candidates, $T(s)$, consists of $\mathbf{0} \in R^n$ together with all of the vectors, v, of unit length for which there exists a sequence $\{s_k\} \subset S$ such that $s_k \to s$ and

$$\lim_{k \to \infty} [(s_k - s)/\|s_k - s\|] = v \tag{11}$$

(e.g., see Cohen and Matthews (1980); these directions are also called 'sequential tangent vectors' and 'tangent vectors' in Hestenes (1975, p. 204)). A direction, $w \in T(s)$, will be called a 'directional equilibrium for $\pi(x)$ at the location $s \in S$' if and only if

$$D_w \pi(s) \geq D_v \pi(s) \quad \forall\, v \in T(s) \tag{12}$$

(where $D_v \pi(s)$ is the directional derivative of $\pi(x)$ in the direction v at the point s). Alternatively, $(w, z) \in T(s)$ will be called a 'directional electoral equilibrium at the location $s \in S$' if and only if

$$\begin{aligned} &D_w \text{Pl}_1(s,s) \geq D_u \text{Pl}_1(s,s) \quad \forall\, u \in T(s), \\ &D_z \text{Pl}_2(s,s) \geq D_v \text{Pl}_2(s,s) \quad \forall\, v \in T(s) \end{aligned} \tag{13}$$

(where $D_u \text{Pl}_1(s,s)$ is the directional derivative of $\text{Pl}_1(\psi_1, s)$ in the direction u at $\psi_1 = s$, and $D_v \text{Pl}_2(s,s)$ is the directional derivative of $\text{Pl}_2(s, \psi_2)$ in the direction v at $\psi_2 = s$).

Corollary 2. *There exists a social utility function $\pi(x)$ on X which rationalizes the directional Nash behavior of candidates and is such that, for any given $S \subset X$, $(w, z) \in T(s) \times T(s)$ is a directional electoral equilibrium at the location $s \in S$ if and only if w and z are directional equilibria for $\pi(x)$ at s.*

$s \in S$ will be called a 'stationary outcome for $\pi(x)$' if and only if

$$D_u \pi(s) \leq D_0 \pi(s) \quad \forall\, u \in T(s) \tag{14}$$

(where, of course, $D_0 \pi(s) = 0$). Alternatively, we will say that there is a 'stationary electoral equilibrium at the location $s \in S$' if and only if $(0,0) \in R^n \times R^n$ is a directional electoral equilibrium at s. As a consequence of Corollary 2 we have the following.

Corollary 3. *There exists a social utility function $\pi(x)$ on X which rationalizes the directional Nash behavior of candidates and is such that, for any given $S \subset X$, there is a stationary electoral equilibrium at the location $s \in S$ if and only if s is a stationary outcome for $\pi(x)$.*

$s \in \text{int}(S)$ will be a 'non-degenerate local maximum' for $\pi(x)$ if and only if s is a stationary outcome for the function $\pi(x)$ and, in addition, the matrix $[\partial^2\pi(x)/\partial x_h \partial x_k]$ (with $h, k = 1, \ldots, n$) is negative definite at $x = s$ (as in Section 5). $s \in \partial(S)$ will be called a 'non-degenerate local maximum' for $\pi(x)$ if and only if (1) s is a regular point in S (as defined in Hestenes (1975, p. 221)), and (2) there are multipliers $\lambda_1, \ldots, \lambda_m$ (with $\lambda_\alpha \geq 0$ for $\alpha = 1, \ldots, p$ and $\lambda_\alpha = 0$ if $g_\alpha(s) < 0$) for which the Lagrangian $F(x) = \pi(x) + \lambda_1 g_1(x) + \cdots + \lambda_m g_m(x)$ is such that (i) $\nabla F(x) = 0$, and (ii) $[\partial^2 F(x)/\partial x_h \partial x_k]$ (with $h, k = 1, \ldots, n$) is negative definite over the set $\{u \in T(s): u \neq 0 \,\&\, D_u \pi(x) = 0\}$ at $x = s$.

If $s \in \text{int}(S)$, we will say that there is a 'non-degenerate' local electoral equilibrium at the location s if and only if there is a stationary equilibrium at the point and the matrices

$$\left[\frac{\partial^2 \text{Pl}_1(\psi_1, s)}{\partial \psi_{1h} \partial \psi_{1k}}\right] \quad \text{and} \quad \left[\frac{\partial^2 \text{Pl}_2(s, \psi_2)}{\partial \psi_{2h} \partial \psi_{2k}}\right] \qquad (h, k = 1, \ldots, n)$$

are negative definite at $\psi_1 = s$ and $\psi_2 = s$, respectively (as in Section 5). For the rest of this definition, let $h_1(x \mid s) = \text{Pl}_1(x, s)$ and $h_2(x \mid s) = \text{Pl}_2(s, x)$. If $s \in \partial(S)$, we will say that there is a 'non-degenerate local electoral equilibrium at the location s' if and only if (1) s is a regular point in S (as defined in Hestenes (1975, p. 221)), and (2) for each $i = 1, 2$ there are multipliers $r_1^i, \ldots, r_m^i$ (with $r_\alpha^i \geq 0$ for $\alpha = 1, \ldots, p$ and $r_\alpha^i = 0$ if $g_\alpha(s) < 0$) for which the Lagrangian $L_i(x) = h_i(x \mid s) + r_1^i g_1(x) + \cdots + r_m^i g_m(x)$ is such that (i) $\nabla L_i(x) = 0$, and (ii) $[\partial^2 L_i(x)/\partial x_h \partial x_k]$ (with $h, k, = 1, \ldots, n$) is negative definite over the set $\{u \in T(s): u \neq 0 \,\&\, D_u h(x \mid s) = 0\}$ at $x = s$.

For a thorough discussion of the concepts used in these definitions and theorems that establish that the resulting points are local maxima, see Hestenes (1975, Chapter 4).

Corollary 4. *There exists a social utility function $\pi(x)$ on X which rationalizes the directional Nash behavior of candidates and is such that, for any given $S \subset X$, there is a non-degenerate local electoral equilibrium at the location $s \in S$ if and only if s is a non-degenerate local maximum for $\pi(x)$.*

Corollaries 3 and 4 extend the most important implication of Theorems 2 and 3 in Coughlin and Nitzan (1981b) to the entire class of societies with probabilistic voting that is smooth, scalable and unbiased. In addition, as with the unconstrained equilibria studied in Section 5, these corollaries also imply the rationalizability of societies' global electoral equilibria whenever there are (correspondingly) sufficient concavity and convexity properties in the candidates' payoff functions for these equilibria to exist.

7. Examples of social utility functions for strategic decisions

In the societies studied in Coughlin and Nitzan (1981a), the Nash Social Welfare Function is a utility function which rationalizes all of the strategic decisions of the candidates that we have studied here. In the societies studied in Coughlin and Nitzan (1981b), the Social Log-likelihood Function has this property. The remainder of this section will be devoted to the problem of constructing an example of a utility function which has this property for every society which has probabilistic voting that is smooth, scalable and unbiased.

$K(\tau) \subset X$ will be used to denote a C^1 path in X from x_0 to x (with τ being a parameter that takes on values in the set $[c,d] \subset R^1$). At any particular point along the path, $K(\tau) \in X$, the marginal expected plurality for candidate 1 for moving in the direction of the path while candidate 2 remains at the point $K(\tau)$ is $F(K(\tau))K'(\tau)$, where

$$F(K(\tau)) = \left[\frac{\partial \mathrm{Pl}_1(\psi_1, K(\tau))}{\partial \psi_{1h}} \right]_{\psi_1 = K(\tau)}. \tag{15}$$

(This function is also used in the proof of Theorem 1. In particular, see (A.1).) Therefore, the path integral

$$\int_K F = \int_{x_0 = K(c)}^{x = K(d)} F(K(\tau))\,\mathrm{d}K \tag{16}$$

gives a cumulative total for the marginal payoffs that are available along the path K (against fixed opponents). Of course, when we have two candidates who are competing with each other, neither of them could accumulate these payoffs (since both of them are capable of making the same calculations). Nonetheless, any such path integral still serves as a measure of the willingness of the voters to move from x_0 to x along the given path.

Since X is simply connected, there is a C^1 path from any $x_0 \in X$ to any other $x \in X$. Therefore, for any given x_0, we can specify a function

$$K(\tau) = \Phi(x; x_0) \tag{17}$$

which assigns a C^1 path, $K(\tau)$, from $x_0 = K(c)$ to $x = K(d)$ for each $x \in X$. For each given x_0 and Φ, (16) specifies a real-valued function

$$\pi(x; x_0, \Phi) = \int_{\Phi(x; x_0)} F \quad \text{on } X \tag{18}$$

(using the proof of Theorem 1 (see Appendix A)). Now using the proof of Theorem 1 again, and Theorem 9.2 of Curtis (1972) we have the following.

Corollary 5. *For any given $x_0 \in X$, the function $\pi(x; x_0, \Phi)$ is independent of the function Φ.*

Therefore, since we always obtain the same function on X once x_0 is specified, we can shorten (18) to $\pi(x;x_0) = \pi(x;x_0,\Phi)$. Using our Theorem 1 and Theorem 9.3 of Curtis (1972), we have the following.

Corollary 6. *For any given $x_0 \in X$, the function $\pi(x;x_0)$ is a utility function which rationalizes the Nash behavior of candidates.*

From this corollary we can also directly conclude that $\pi(x;x_0)$ rationalizes the locations where Nash points occur. In addition, by using this corollary along with Theorem 9.3 of Curtis (1972) and Lemma A.1 it also follows that $\pi(x;x_0)$ rationalizes the non-degenerate local electoral equilibria in the societies under consideration.

Of course, it is not being claimed that the social utility functions which can be constructed in this way have any particular appeal from a normative point of view – just that they rationalize the strategic (directional) decisions of candidates at the various possible starting locations in X.

8. Conclusion

This paper has studied societies which have probabilistic voting that is smooth, scalable and unbiased. Its results have established that, in such societies, the decisions of vote-seeking candidates who start at a common location (such as the status quo for the society's policies and/or the same allocations of campaign resources) contain implicit rationality properties. In particular, it has shown that, in every such society, there exist social utility functions which simultaneously rationalize the directional Nash behavior of candidates, the stationary electoral equilibria, and the non-degenerate local electoral equilibria which can occur at these locations – both for constrained and unconstrained sets of possible candidate locations. In addition, an example of such a utility function (which occurs in all of the societies under consideration) has been provided.

Appendix A

Proof of Theorem 1. To begin with, we will let $F(x)$ denote the vector field that assigns to each $x \in X$ the gradient of candidate 1's payoff function at that location (under the Nash assumption that his rival's position is fixed). That is, we will let

$$F(x) = \left[\frac{\partial \mathrm{Pl}_1(\psi_1, x)}{\partial \psi_{1h}} \right]_{\psi_1 = x} \quad \text{for each } x \in X \tag{A.1}$$

(*remark*: this function also appears in (16)). Following the discussion in Section 3 and at the beginning of Section 4, it is enough to show that the premise of the

theorem implies the existence of a real-valued, totally differentiable function, $\pi(x)$, on X which has

$$\left[\frac{\partial \pi(x)}{\partial x_h}\right]_{x=z} = F(z) \quad \text{at every } z \in X. \tag{A.2}$$

That is, that $F(x)$ is a gradient field which has a totally differentiable potential function. We will use Theorem 9.4 of Curtis (1972) to establish that $F(x)$ is a gradient field. Then, when this is done, we will turn to the differentiability properties of its potential functions.

In order to use Theorem 9.4 of Curtis (1972) we will first have to show that $F(x)$ is a continuously differentiable function. That is, we must show that each component function, $F_h(x)$, in $F(x)$ is continuously differentiable (for $h = 1, \ldots, n$). This will be accomplished by using the Chain Rule (as stated in Theorem 6.12 of Curtis (1972)). $F(x)$ can be translated into Curtis' notation as follows. First, let $g: X \to X^2$ be defined by

$$g_i(x) = \begin{cases} x_i & \text{if } i = 1, \ldots, n. \\ x_{i-n} & \text{if } i = n+1, \ldots, 2n. \end{cases} \tag{A.3}$$

Then, let $f: X^2 \to R^n$ be defined by

$$f_h(g(x)) = \partial \mathrm{Pl}_1(g(x)) \,/\, \partial \psi_{1h} \tag{A.4}$$

where, by (A.3),

$$\frac{\partial \mathrm{Pl}_1(g_1(x), \ldots, g_n(x);\, g_{n+1}(x), \ldots, g_{2n}(x))}{\partial \psi_{1h}} = \frac{\partial \mathrm{Pl}_1(x, x)}{\partial \psi_{1h}}. \tag{A.5}$$

This gives us $F(x) = f \circ g(x)$. Now, since the probabilistic voting is smooth, each f_h is a differentiable function of its components at each possible $g(x)$. Furthermore, by their construction, each $g_i(x)$ is a differentiable function of its components at each possible $x \in X$. Therefore, by the Chain Rule, $F(x) = f \circ g(x)$ is differentiable at every $x \in X$. In addition, at any given $z \in X$, the partial derivatives of $F(x)$ are given by

$$\begin{bmatrix} \frac{\partial F_1(z)}{\partial x_1} & \cdots & \frac{\partial F_1(z)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n(z)}{\partial x_1} & \cdots & \frac{\partial F_n(z)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(g(z))}{\partial g_1} & \cdots & \frac{\partial f_1(g(z))}{\partial g_{2n}} \\ \vdots & & \vdots \\ \frac{\partial f_n(g(z))}{\partial g_1} & \cdots & \frac{\partial f_n(g(z))}{\partial g_{2n}} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1(z)}{\partial x_1} & \cdots & \frac{\partial g_1(z)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_{2n}(z)}{\partial x_1} & \cdots & \frac{\partial g_{2n}(z)}{\partial x_n} \end{bmatrix}. \tag{A.6}$$

That is,

$$\begin{bmatrix} \frac{\partial F_1(z)}{\partial x_1} & \cdots & \frac{\partial F_1(z)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n(z)}{\partial x_1} & \cdots & \frac{\partial F_n(z)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{11}^2} & \cdots & \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2n} \partial \psi_{11}} \\ \vdots & & \vdots \\ \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{11} \partial \psi_{1n}} & \cdots & \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2n} \partial \psi_{1n}} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1(z)}{\partial x_1} & \cdots & \frac{\partial g_1(z)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_{2n}(z)}{\partial x_1} & \cdots & \frac{\partial g_{2n}(z)}{\partial x_n} \end{bmatrix}. \tag{A.7}$$

where

$$\frac{\partial g_j(z)}{\partial x_k} = \begin{cases} 1 & \text{if } j=k \text{ or } j=n+k, \\ 0 & \text{otherwise.} \end{cases} \tag{A.8}$$

Therefore, for each pair h, $k \in \{1, \ldots, n\}$, we have

$$\frac{\partial F_h(z)}{\partial x_k} = \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1k} \partial \psi_{1h}} + \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}}. \tag{A.9}$$

Having smooth probabilistic voting implies that, whenever $x=y=z$, the function $\mathrm{Pl}_1(x,y)$ is a C^2 function of its arguments in some neighborhood of $(x,y)=(z,z)$. This means (among other things) that, for each pair h, $k \in \{1, \ldots, n\}$ and location $z \in X$, both $\partial^2 \mathrm{Pl}_1(x,y)/\partial \psi_{1k} \partial \psi_{1h}$ and $\partial^2 \mathrm{Pl}_1(x,y)/\partial \psi_{2k} \partial \psi_{1h}$ are continuous functions of (x,y) in some neighborhood of $(x,y)=(z,z)$. Therefore, in particular, $\partial^2 \mathrm{Pl}_1(z,z)/\partial \psi_{1k} \partial \psi_{1h}$ and $\partial^2 \mathrm{Pl}_1(z,z)/\partial \psi_{2k} \partial \psi_{1k}$ are continuous functions of z at every $z \in X$. Therefore, by (A.9), each $\partial F_h(z)/\partial x_k$ is a continuous function of z at every $z \in X$.

Now that we know that $F(x)$ is a continuously differentiable vector field, Theorem 9.4 of Curtis (1972) implies that there is a real-valued differentiable function $\pi(x)$ for which (A.2) is satisfied if

$$F'(z) = \left[\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1k} \partial \psi_{1h}} + \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}} \right] \quad (h,k = 1, \ldots, n) \tag{A.10}$$

is a symmetric matrix. We will establish that this is indeed the case through the following argument. Consider any entry in the matrix that has $h \neq k$. By Young's theorem (Theorem 6.5 of Curtis (1972)) together with Corollary 5.9 of Bartle (1965), we have

$$\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1k} \partial \psi_{1h}} = \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1h} \partial \psi_{1k}}. \tag{A.11}$$

Therefore, symmetry will follow if we can show that

$$\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}} = \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2h} \partial \psi_{1k}}. \tag{A.12}$$

By (2), (3), the Chain Rule and Corollary 5.9 of Bartle (1965), when $x=y$ we have

$$\frac{\partial \mathrm{Pl}_1(x,y)}{\partial \psi_{1h}} =$$

$$= \frac{\partial}{\partial \psi_{1h}} \left[\int_A \bar{P}_a^1(G_a(\psi_1), G_a(\psi_2))\,\mathrm{d}\mu(a) - \int_A \bar{P}_a^2(G_a(\psi_1), G_a(\psi_2))\,\mathrm{d}\mu(a) \right]_{\substack{\psi_1 = x \\ \psi_2 = y}}$$

$$= \left[\int_A \frac{\partial}{\partial \psi_{1h}} \bar{P}_a^1(G_a(\psi_1), G_a(\psi_2))\,\mathrm{d}\mu(a) \right.$$

$$\left. - \int_A \frac{\partial}{\partial \psi_{1h}} \bar{P}_a^2(G_a(\psi_1), G_a(\psi_2))\,\mathrm{d}\mu(a) \right]_{\substack{\psi_1 = x \\ \psi_2 = y}}$$

$$= \int_A \frac{\partial \bar{P}_a^1(G_a(x), G_a(y))}{\partial g_a(x)} \frac{\partial G_a(x)}{\partial x_h} d\mu(a)$$

$$- \int_A \frac{\partial \bar{P}_a^2(G_a(x), G_a(y))}{\partial G_a(x)} \frac{\partial G_a(x)}{\partial x_h} d\mu(a). \tag{A.13}$$

Therefore, by Corollary 5.9 of Bartle (1965) and the Chain Rule again, when $x = y$ we have

$$\frac{\partial^2 \mathrm{Pl}_1(x, y)}{\partial \psi_{2k} \partial \psi_{1h}} = \int_A \frac{\partial^2 \bar{P}_a^1(G_a(x), G_a(y))}{\partial G_a(x) \partial G_a(y)} \frac{\partial G_a(y)}{\partial y_k} \frac{\partial G_a(x)}{\partial x_h} d\mu(a)$$

$$- \int_A \frac{\partial^2 \bar{P}_a^2(G_a(x), G_a(y))}{\partial G_a(x) \partial G_a(y)} \frac{\partial G_a(y)}{\partial y_k} \frac{\partial G_a(x)}{\partial x_h} d\mu(a). \tag{A.14}$$

Similarly, whenever $x = y$ we have

$$\frac{\partial^2 \mathrm{Pl}_1(x, y)}{\partial \psi_{2h} \partial \psi_{1k}} = \int_A \frac{\partial^2 \bar{P}_a^1(G_a(x), G_a(y))}{\partial G_a(x) \partial G_a(y)} \frac{\partial G_a(y)}{\partial y_h} \frac{G_a(x)}{\partial x_k} d\mu(a)$$

$$- \int_A \frac{\partial^2 \bar{P}_a^2(G_a(x), G_a(y))}{\partial G_a(x), G_a(y)} \frac{\partial G_a(y)}{\partial y_h} \frac{\partial G_a(x)}{\partial x_k} d\mu(a). \tag{A.15}$$

But, at $x = y = z$, we must have

$$\frac{\partial G_a(y)}{\partial y_k} \frac{\partial G_a(x)}{\partial x_h}\bigg|_{\substack{y=z \\ x=z}} = \frac{\partial G_a(x)}{\partial x_k} \frac{\partial g_a(y)}{\partial y_h}\bigg|_{\substack{x=z \\ y=z}}. \tag{A.16}$$

Therefore, by (A.14) - (A.16) we have (A.12). That is, $F'(z)$ is symmetric at every $z \in X$.

Now, the only thing that remains is to show that at least one of the potential functions for $F(z)$ is totally differentiable. Since we have smooth probabilistic voting, whenever $x = y = z \in X$, each $\partial \mathrm{Pl}_1(x, y)/\partial \psi_{1h}$ $(h = 1, \ldots, n)$ is a continuous function of (x, y) in some neighborhood of $(x, y) = (z, z)$. Therefore, each $\mathrm{Pl}_1(z, z)/\partial \psi_{1h}$ $(h = 1, \ldots, n)$ is a C^1 function of (z, z) at every $z \in X$. Therefore, for any potential function $\pi(x)$, (A.2) implies that each $\partial \pi(z)/\partial \psi_{1h}$ must be a continuous function of z at every $z \in X$. That is, every potential function for $F(z)$ must be continuously differentiable at every $z \in X$. Since this is the case, each $\pi(x)$ is also totally differentiable (e.g., see Theorem 6.8 of Curtis (1972)). □

Proof of Theorem 2. Theorem 2 follows from Theorem 1 together with the following lemma.

Lemma A.1. (*Taking the premise of the theorem as given*) *every potential function for* $F(x)$, $\pi(x)$, (*see* (A.1)) *has*

$$\left[\frac{\partial^2 \pi(z)}{\partial x_h \partial x_k}\right] = \left[\frac{\partial^2 \mathrm{Pl}_1(z, z)}{\partial \psi_{1h} \partial \psi_{1k}}\right] \quad (h, k \in \{1, \ldots, n\})$$

at every $z \in X$.

Proof. In the proof of Theorem 1 we showed that $F(x)$ is continuously differentiable. Therefore (as in Curtis (1972, pp. 348–349)), each potential function for $F(x)$, $\pi(x)$ must be twice continuously differentiable. Furthermore, we must have

$$\frac{\partial^2 \pi(z)}{\partial x_h \partial x_k} = \frac{\partial F_h(z)}{\partial x_k} \tag{A.17}$$

for each pair of indices h, $k \in \{1, \ldots, n\}$. Therefore, by (A.9),

$$\frac{\partial^2 \pi(z)}{\partial x_h \partial x_k} = \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1k} \partial \psi_{1h}} + \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}} \tag{A.18}$$

for each pair of indices $h,k \in \{1, \ldots, n\}$. Hence, the lemma will follow if we can show that

$$\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}} = 0 \tag{A.19}$$

for each pair of indices h, $k \in \{1, \ldots, n\}$ at each $z \in X$. This will be done in three steps.

Step 1. By (3) and (7), $\mathrm{Pl}_1(x,y) = -\mathrm{Pl}_1(y,x)$. Therefore,

$$\frac{\partial^2 \mathrm{Pl}_1(x,y)}{\partial \psi_{1h} \partial \psi_{2k}} = -\frac{\partial^2 \mathrm{Pl}_1(x,y)}{\partial \psi_{2h} \partial \psi_{1k}} \tag{A.20}$$

for each pair of indices, h, $k \in \{1, \ldots, n\}$ at each $(x,y) \in X^2$. Therefore,

$$\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1h} \partial \psi_{2k}} = -\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2h} \partial \psi_{1k}} \tag{A.21}$$

for each pair of indices $h, k \in \{1, \ldots, n\}$ at each $z \in X$.

Step 2. In the proof of Theorem 1 we have already established that

$$\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}} = \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2h} \partial \psi_{1k}} \tag{A.22}$$

for each pair of indices h, $k \in \{1, \ldots, n\}$ at each $z \in X$. Therefore, by (A.21),

$$\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}} = -\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1h} \partial \psi_{2k}} \tag{A.23}$$

for each pair of indices h, $k \in \{1, \ldots, n\}$ at each $z \in X$.

Step 3. Young's Theorem (Theorem 6.5 of Curtis (1972)) implies

$$\frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{2k} \partial \psi_{1h}} = \frac{\partial^2 \mathrm{Pl}_1(z,z)}{\partial \psi_{1h} \partial \psi_{2k}}$$

for each pair of indices h, $k \in \{1, \ldots, n\}$ at each $z \in X$. Therefore, (A.19) follows. □

Proof of Corollary 2. The proof of Theorem 1 has established that there exists a totally differentiable function $\pi(x)$, which satisfies (A.2). We will show that any

such function satisfies the corollary's requirement. To begin with, if a location is in the interior of S, then the requirement follows directly (as in Theorem 1). Alternatively, if a location s is on the boundary of S, then the following arguments apply. First, the set of feasible directions for the candidates is identical to the set of feasible directions for a decision with $\pi(x)$ (being, in both cases, $T(s)$). Second, since $\pi(x)$ is totally differentiable at every $x \in X$ and $\mathrm{Pl}_1(x, y)$ and $\mathrm{Pl}_2(y, x)$ are totally differentiable at every $x \in X$ where $x = y$, all three of these functions have directional derivatives for each of the non-zero directions in $T(s)$. Finally, since (A.2) is satisfied, each of these three directional derivatives must be equal at every non-zero direction in $T(s)$. Therefore, the corollary's requirement must be satisfied. □

Proof of Corollary 4. This corollary follows from (i) the differentiability properties, the function $\pi(x)$, the candidates' payoff functions, and the constraints that define S, (ii) Lemma A.1, together with (iii) the fact that the set of feasible directions is the same in each of the decision problems (using reasoning similar to that which appears in the proof of Corollary 3). □

Notes

Note 1. For instance, see Schofield (1978), Cohen and Matthews (1980), Matthews (1979, 1980), McKelvey, Ordeshook and Ungar (1980) and the related references which they cite.

Note 2. For additional references on the topic of collective decision making with probabilistic voting, see Intriligator (1973, 1979, 1982), McKelvey (1975), Slutsky (1975), Denzau and Kats (1977), Kramer (1978a) and the related articles which they cite.

Note 3. A set $X \subset R^n$ is 'star-shaped' if there is a point $s \in X$ such that for each $x \in X$ the closed line segment joining x and s lies entirely in X (from p. 354 in Curtis (1972)). This, of course, generalizes the standard assumption that a candidate's possible locations form a convex set.

Note 4. In Hinich and Ordeshook (1969, 1971), Hinich, Ledyard and Ordeshook (1972, 1973) and Hinich (1977, 1978), $G_a(x) = U_a(x)$. In Coughlin and Nitzan (1981a, 1981b), $G_a(x) = f(x;a)$ (or $f_a(x)$) and

$$\bar{P}_a^i(G_a(\psi_1), G_a(\psi_2)) = G_a(\psi_i)/(G_a(\psi_i) + G_a(\psi_j)) \quad \text{for } i, j \in C \text{ and } i \neq j.$$

Note 5. To identify the special cases of scalability which occur in Brams and Davis (1973, 1974) and Brams (1978, Chapter 3), start by letting a be an index for a *state* ($a = 1, \ldots, n$). Let x_a be the amount of money spent by candidate 1 in *state a*. $x = (x_1, \ldots, x_n) \in R^n$ then specifies a strategy for candidate 1 (i.e., the amounts to be spent in the various *states*). Similarly, let $y = (y_1, \ldots, y_n) \in R^n$ denote a (resource allocation) strategy for candidate 2. Let $G_a(x) = x_a$ (i.e., $G_a(x)$ is simply the ath component of x). This implies $G_a(y) = y_a$ as well. Then, their popular vote model follows when

$$\bar{P}_a^1(x_a, y_a) = x_a/(x_a + y_a) \quad \text{and} \quad \bar{P}_a^2(x_a, y_a) = y_a/(x_a + y_a).$$

Alternatively, their electoral vote model follows when

$$\bar{P}_a^1(x_a, y_a) = \sum_{k=(n_a+1)/2}^{n_a} \binom{n_a}{k} \left(\frac{x_a}{x_a + y_a}\right)^k \left(\frac{y_a}{x_a + y_a}\right)^{n_a - k}$$

and $\bar{P}_a^2(x_a, y_a) = 1 - \bar{P}_a^1(x_a, y_a)$. In addition, in their popular vote model, $\mu(a) = n_a$, while in their electoral college model $\mu(a)$ is the number of electoral votes from state a.

Note 6. For the model in Shane (1977, Section 3) we start out the same way as for the Brams-Davis (1973, 1974) model, but then let

$$\bar{P}_a^1(x_a, y_a) = P_a + P_a q_a[(x_a - y_a)/(x_a + y_a)]$$

and

$$\bar{P}_a^2(x_a, y_a) = 1 - \bar{P}_a^1(x_a, y_a).$$

Note 7. For the Brams-Davis (1982) model, let a be an index for a *primary* ($a = 1, \ldots, n$) Let x_a be the amount of money spent by candidate 1 in *primary* a. $x = (x_1, \ldots, x_n) \in R^n$ then specifies a strategy for candidate 1 (i.e., the amounts to be spent in the various primaries). Similarly, let $(y_1, \ldots, y_n)$ denote a (resource allocation) strategy for candidate 2. They assume (see Brams and Davis (1982, p. 378)) that

$$P_a^1(x, y) = \frac{P_{a-1}^1 x_a}{P_{a-1}^1 x_a + (1 - P_{a-1}^1) y_a}$$

for each a (where $P_0^1 = \frac{1}{2}$). This directly implies

$$P_1^1 = x_1/(x_1 + y_1).$$

Now, suppose that for a given $t \in \{1, \ldots, n-1\}$.

$$P_t^1 = \frac{x_1 x_2 \cdots x_t}{x_1 \cdots x_t + y_1 \cdots y_t}.$$

Then, substituting for P_t^1 in the equation for P_{t+1}^1 and cancelling the common denominator we have

$$P_{t+1}^1(x, y) = \frac{x_1 \cdots x_{t+1}}{x_1 \cdots x_{t+1} + y_1 \cdots y_{t+1}}.$$

Therefore, by induction,

$$P_a^1 = \frac{x_1 \cdots x_a}{x_1 \cdots x_a + y_1 \cdots y_a}$$

for each a. This implies that we have scalability – with $G_a(x) = x_1 \cdots x_a$ (and, hence, $G_a(y) = y_1 \cdots y_a$).

$$\bar{P}_a^1(G_a(x), G_a(y)) = \frac{G_a(x)}{G_a(x) + G_a(y)}$$

and

$$\bar{P}_a^2(G_a(x), G_a(y)) = 1 - \bar{P}_a^1(G_a(x), G_a(y)).$$

In addition, in their primary model, $\mu(a) = v_a =$ the number of delegates being selected in primary a.

Note 8. The term 'smooth' has been used in different references to mean different things (including C^1, C^2 and C^∞). However, since an exact definition of its meaning here has been provided, there should be no confusion.

Note 9. For an analysis of the same basic rationality requirement in the context of consumer decision making, see (for instance) Hurwicz and Uzawa (1971), Katzner (1970) and Varian (1978).

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